Rolle's theorem

Let f be continuous on $[a,b]$ and differentiable on (a,b), with $a<b$. If $f(a)=f(b)$, then there is a point $x_0 \in (a,b)$ such that $f'(x_0)=0$.

Proof

Since f continuous on $[a,b]$, f attains both its max and min on $[a,b]$. Suppose $f(a)=f(b)$ is both max and min, then f is constant function $f(x)=f(a)$. So $f'(x)=0$ for all $x \in (a,b)$ and for $x_0 \in (a,b)$ in particular. Alternatively suppose, WLOG, max is different from $f(a)$. Since max is not at endpoint and $f'(x)$ exists throughout (a,b), the maximum value of f is attained at a point $x_0 \in (a,b)$ for which $f'(x_0)=0$, by a previous theorem.

□

MEAN Value thm (MVT)

Let f be continuous on $[a,b]$ where $a<b$ and differentiable on (a,b). Then there exists $x_0 \in (a,b)$ such that

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}.$$

Proof

Define function s on $[a,b]$ by

$$s(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

s is obviously continuous on $[a,b]$ and differentiable on (a,b). Note that $s(a)=s(b)$, since both equal zero. The Conditions of Rolle's theorem are met, so there exists $x_0 \in (a,b)$ such that $s'(x_0)=0$. Now, $s'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

Therefore,

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

for some $x_0 \in (a,b)$

□